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P.I. Algebras with Hopf Algebra Actions*

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If A is a p.i. algebra in characteristic zero with action from a finite-dimensional semisimple Hopf algebra H, then A has a nilpotent H-ideal N such that A/N will be H-verbally semiprime. Every H-verbally semiprime algebra is H-p.i. equivalent to a direct sum of H-verbally prime algebras. In the case of a finite group action or a grading by an abelian group, we show that the sum can be taken to be finite. In the case of an action by a finite cyclic group G, we classify all G-p.i. algebras, up to equivalence. This paper generalizes the work of A. R. Kerner (1985, *Math. USSR Izv*. **25**).

Key Words: p.i. algebra; Hopf algebra action; group action; group grading; verbally prime; verbally semiprime.

In [3] Kemer proved a basic classification theorem for p.i. algebras in characteristic zero. We recall his results briefly: If A is a p.i. algebra over a characteristic zero field, then A has a nilpotent ideal I such that A/I is *verbally semiprime*. This means that if $f(x_1, \dots, x_n)$ is not an identity for A/I, then neither is $f(x_1, \dots, x_n)f(x_{n+1}, \dots, x_{2n})$. Next, if A is a verbally semiprime algebra then it is p.i. equivalent to a finite direct sum of *verbally prime* algebras. Verbally prime means that if $f(x_1, \dots, x_n)$ is not an identity and $g(x_1, \dots, x_m)$ is not an identity then $f(x_1, \dots, x_n) \times g(x_{n+1}, \dots, x_{n+m})$ is not an identity. Finally, Kemer classifies all verbally prime algebras, again, up to p.i. equivalence. Every verbally prime p.i. algebra is equivalent to either the $n \times n$ matrices over the field, for some n; or, to $n \times n$ matrices over the Grassmann algebra, for some n; or to $M_{k,l}$, for some k, l, which we define below. It turns out that the $n \times n$ matrices over the field are p.i. equivalent to $M_{n,0}$ so there are really two families of verbally prime p.i. algebras, $M_n(E)$ and $M_{k,l}$.

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A key ingredient in Kemer's work is the use of $\mathbf{Z}/2\mathbf{Z}$-graded algebras. The Grassmann algebra, E, is an important example of a $\mathbf{Z}/2\mathbf{Z}$-graded algebra. Throughout this paper the Grassmann algebra E will be over an unspecified infinite-dimensional vector space. The degree zero part E_0 is spanned by products of even numbers of vectors and the degree one part E_1 is spanned by odd numbers. Note that E_0 is the center of E and that elements of E_1 anticommute with each other. Any $\mathbf{Z}/2\mathbf{Z}$-graded algebra with these two properties is called supercommutative. Given any $\mathbf{Z}/2\mathbf{Z}$-graded algebra A he defines A^* to be $A_0 \otimes E_0 \oplus A_1 \otimes E_1$. This operation is an involution on graded varieties in the sense that if A and B satisfy the same $\mathbf{Z}/2\mathbf{Z}$-graded identities, then so do A^* and B^*; and that A satisfies the same graded identities as A^{**}. The reader should note that, by Kemer's theorem, every verbally prime algebra is p.i. equivalent to some A^* or $A \otimes E$, where A is simple. Kemer has a generalization of the notion of graded algebras we will describe later with respect to which all of the verbally prime algebras are of the form A^* for A graded simple.

It is our goal in this paper to generalize this theory to p.i. algebras with an action by a finite-dimensional semisimple Hopf algebra H, which we take to be fixed. One may define H-polynomials and H-polynomial identities, and use this definition to define H-verbally prime and H-verbally semiprime. Here then is our main theorem:

THEOREM. (1) *Let A be any p.i. algebra in characteristic zero with action from the finite-dimensional semisimple Hopf algebra H. Then there is a nilpotent H-ideal I such that A/I is H-verbally semiprime.*

(2) *Let A be H-verbally semiprime. Then A is H-p.i. equivalent to a direct sum of H-verbally prime algebras.*

(3) *Every H-verbally prime algebra is H-p.i. equivalent to either some B^*, where B is H-prime, or some $B \otimes E$, where B is H-prime.*

In part (2) of the theorem we are not able to prove that the direct sum will be finite in general. We are able to prove it in the special cases in which H is a group algebra where the group elements acts as automorphisms or H is dual of a group algebra of an abelian group and the action corresponds to a grading by the group. The sticking point in the proof is the need to show that a given finite-dimensional algebra A accepts only finitely many H-actions, up to automorphism. We are happy to thank R. Guralnick and P. Slodowy for helping us with the group case. In these special cases we can also strengthen part (3) of the above theorem and show that the algebras B which occur must be H-simple and finite-dimensional.

Note that an algebra with H-action may satisfy H-identities but not be p.i. Our theorem does not apply to these algebras. We close the paper by classifying algebras with action by a cyclic group and algebras with a $\mathbf{Z}/2\mathbf{Z}$-grading.

1. DEFINITIONS

We will be working throughout with p.i. algebras A with an action from the fixed, finite-dimensional semisimple Hopf algebra H, all over the characteristic zero field F. The first author discussed the H-identities of these algebras in [1]. The idea is this: Given a set of variables X one may form $F\langle X|H\rangle$, the free F-algebra with H-action on the set X. $F\langle X|H\rangle$ has the universal property that if A is an H-algebra, then any set theoretic map $X \to A$ extends to an H-algebra map $F\langle X|H\rangle \to A$. Elements of the universal algebra are H-polynomials, and an H-polynomial which vanished under all H-homomorphisms (substitutions) from $F\langle X|H\rangle$ to A would be an H-identity for A. For example, let A be $\mathbf{Z}/2\mathbf{Z}$-graded and let π_i, $i = 0, 1$, be the projections onto the homogeneous parts. Then A is supercommutative if it satisfies the three identities

$$f(x, y) = \pi_0(x)\pi_0(y) - \pi_0(y)\pi_0(x),$$
$$g(x, y) = \pi_0(x)\pi_1(y) - \pi_1(y)\pi_0(x)$$

and

$$h(x, y) = \pi_1(x)\pi_1(y) + \pi_1(y)\pi_1(x).$$

In order to imitate Kemer's constructions in the p.i. case, we now generalize to algebras B with H-action and with $\mathbf{Z}/2\mathbf{Z}$-grading, such that the action preserves the grading. Graded H-polynomials are elements of the free, graded H-algebra $F\langle X \cup Y|H\rangle$ in which letters from X are degree zero and letters from Y are degree one. And, a graded H-identity for B would be such a polynomial $f(x_1, \dots, x_n, y_1, \dots, y_n)$ which vanished under all graded substitutions into B. The ideal of identities for an algebra is called an H-T-ideal. It will be invariant under all H-homomorphisms from $F\langle X \cup Y|H\rangle$ to itself. An equivalent point of view which is also useful is to consider B as an algebra with action from the Hopf algebra $(\mathbf{Z}/2\mathbf{Z}) \times H$, where $(\mathbf{Z}/2\mathbf{Z}) \times H$ denotes the tensor product of H with the group algebra $F(\mathbf{Z}/2\mathbf{Z})$. From this point of view a $(\mathbf{Z}/2\mathbf{Z}) \times H$-polynomial would be an element of $F\langle Z|(\mathbf{Z}/2\mathbf{Z}) \times H\rangle$, where Z is an ungraded alphabet. The degree is encoded instead in the action. So, for example, instead of $h_1(x_1)h_2(y_2)$ we would have $h_1(\pi_0(z_1))h_2(\pi_1(z_2))$. We will alternate freely between these two points of view.

As usual in p.i. theory, the multilinear polynomials play a special role. For our purposes, the more useful definition of multilinear would be with respect to the z's rather than the x's and y's. So, a polynomial $f(x_1, \dots, x_n, y_1, \dots, y_n)$ would be multilinear if for all $\alpha_1, \dots, \alpha_n \in F$,

$$f(\alpha_1 x_1, \dots, \alpha_n x_n, \alpha_1 y_1, \dots, \alpha_n y_n) = \alpha_1 \cdots \alpha_n f(x_1, \dots, x_n, y_1, \dots, y_n).$$

We denote the space of $\mathbf{Z}/2\mathbf{Z}$-graded, multilinear, degree n H-polynomials as $V_n(Z|H)$ and the subspace of identities for B as $V_n(B|H)$. As usual, two algebras will satisfy the same graded, H-identities if and only if they satisfy the same graded, multilinear H-identities. This implies in particular that if K is a field extension of F that A and $A \otimes_F K$ satisfy the same identities. There is a useful connection with representation theory which we now describe.

There is an identification between multilinear polynomials of degree n and elements of the wreath product $(\mathbf{Z}/2\mathbf{Z} \times H) \sim S_n$. (Unfortunately the same symbol $\sim$ is used for both p.i. equivalence and wreath products. Hopefully it will be obvious which one is meant from the context.) The identification is given by: If $\epsilon_1, \dots, \epsilon_n \in \mathbf{Z}/2\mathbf{Z}, h_1, \dots, h_n \in H$ and $\sigma \in S_n$, then $((\epsilon_1, h_1), \dots, (\epsilon_n, h_n); \sigma) \in (\mathbf{Z}/2\mathbf{Z} \times H) \sim S_n$ is identified with $h_{\sigma(1)}(u_{\sigma(1)}) \cdots h_{\sigma(n)}(u_{\sigma(n)})$, where u_i equals x_i or y_i, according to whether ϵ_i is 0 or 1. This identification gives an action of $(\mathbf{Z}/2\mathbf{Z} \times H) \sim S_n$ on $V_n(Z|H)$ corresponding to multiplication on the left with the property that $((\epsilon_1, h_1), \dots, (\epsilon_n, h_n); \sigma) \in (\mathbf{Z}/2\mathbf{Z} \times H) \sim S_n$ times $f(u_1, \dots, u_n)$ equals zero unless the degree of u_i is $\epsilon_{\sigma(i)}$ for all i, in which case it equals $f(h_{\sigma(1)}(u_{\sigma(1)}), \dots, h_{\sigma(n)}(u_{\sigma(n)}))$; i.e., it acts by the graded substitution $u_i \mapsto h_{\sigma(i)}(u_{\sigma(i)})$. In particular, $V_n(B|H)$ will be submodule. By [5], $(\mathbf{Z}/2\mathbf{Z} \times H) \sim S_n$ is semisimple, and so we may write

$$V_n(Z|H) = V_n(B|H) \oplus Q_n(B|H),$$

over $(\mathbf{Z}/2\mathbf{Z} \times H) \sim S_n$. So we may consider $V_n(B|H)$ and $Q_n(B|H)$ as either submodules of $V_n(Z|H)$ or as left ideals of $(\mathbf{Z}/2\mathbf{Z} \times H) \sim S_n$.

2. FINITE GENERATION

Given a $\mathbf{Z}/2\mathbf{Z}$-graded algebra B we define B^* to be $B_0 \otimes E_0 \oplus B_1 \otimes E_1$, where E is the infinite-dimensional Grassmann algebra with its usual $\mathbf{Z}/2\mathbf{Z}$-grading. Note that if B has an H-action then B^* will also carry an H-action, with H acting trivially on E. Let us write $B_1 \sim B_2$ if B_1 and B_2 satisfy the same graded H-identities. Then the $*$-operation on algebras has the properties (1) $B_1 \sim B_2$ if and only if $B_1^* \sim B_2^*$ and (2) $B \sim B^{**}$ for all B. There is also a corresponding involution on multilinear polynomials.

Given a multilinear, degree n polynomial of the form $f(x_{i_1}, \dots, x_{i_a}, y_{j_1}, \dots, y_{j_b})$, where $a + b = n$ choose homogeneous degree one elements of the Grassmann algebra, $e_{j_1}, \dots, e_{j_b}$ and define the multilinear polynomial f^* via

$$f(x_{i_1}, \dots, x_{i_a}, e_{j_1} y_{j_1}, \dots, e_{j_b} y_{j_b}) = f^*(x_{i_1}, \dots, x_{i_a}, y_{j_1}, \dots, y_{j_b}) e_{j_1} \cdots e_{j_b},$$

where the e's commute with the x's and y's. This operation may be extended to all of the $V_n(Z|H)$ by linearity. Note that for any B, f will be an identity for B if and only if f^* is an identity for B^*.

Let A be any (ungraded) p.i. algebra with H-action. Following Kemer we may alter the definition of a $\mathbf{Z}/2\mathbf{Z}$-grading so as to allow the possibility that the degree 0 part and the degree 1 part intersect. Then grade A via $A = A_0 = A_1$. Alternately, we could set $A = A_0$ and $At = A_1$, where t is a central, degree one element with $t^2 = 1$. In any event, a polynomial $f(z_1, \dots, z_n)$ is a graded identity for A if and only if it is an identity for A. Moreover, $A^* = A \otimes E$, with the grading coming from E. Our main technical result is that A^* can be finitely represented, namely, that there exists a finitely generated, graded algebra B such that $B \sim A^*$.

Given k and l, let $U_{k,l}(A)$ be the generic algebra for A as an H-algebra on k degree 0 generators and l degree one generators. So, $U_{k,l}(A)$ would be the free H-algebra on $\{x_1, \dots, x_k, y_1, \dots, y_l\}$ modulo the graded H-identities of A.

LEMMA 1. *Let A be a p.i. algebra with H-action and grade A via $A = A_0 = A_1$. Then for large enough k and l, $A^* \sim U_{k,l}(A^*)$ as (ungraded) H-p.i. algebras.*

Proof. In order to prove this lemma we need to show that if a multilinear degree n polynomial α is not an identity for $U_{(k,l)}(A^*)$ then it is not an identity for A^*. Let W_n be the space of homogeneous degree n polynomials in $x_1, \dots, x_k, y_1, \dots, y_l$. There is an action of $(\mathbf{Z}/2\mathbf{Z} \times H) \sim S_n$ on the right of W_n given by

$$(w_1 \cdots w_n) \cdot f(z_1, \dots, z_n) = f(w_1, \dots, w_n),$$

if w_i and z_i have the same degree for all i, and zero otherwise. A standard linearization-specialization argument shows that for any algebra B

$$W_n V_n(B|H) = \text{the identities of } B \text{ in } W_n.$$

Moreover, from the semisimplicity of $(\mathbf{Z}/2\mathbf{Z} \times H) \sim S_n$ it is not hard to see that

$$W_n = W_n V_n(B|H) \oplus Q_n(B|H).$$

This is because $V_n(B|H) = V_n(Z|H)e$ for an idempotent e in $(\mathbf{Z}/2\mathbf{Z} \times H) \sim S_n \equiv V_n(Z|H)$ and $Q_n(B|H) = V_n(Z|H)f$, where $f = 1 - e$ is an orthogonal idempotent. Hence, $W_n V_n(B|H) = W_n e$ and $W_n Q_n(B|H) = W_n f$. But, if $we = w'f$, then $we = we^2 = w'fe = 0$.

Now, given a p.i. algebra A with H-action, we know from [1] that there exists a k and l such that the cocharacter of A is contained in the $k \times l$ hook, $H(k, l) =$ the set of all partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ such that $\lambda_{k+1} \leq l$. Here is what this means: The algebra $(\mathbf{Z}/2\mathbf{Z} \times H) \sim S_n$ can be written as a direct sum of two sided ideals

$$(\mathbf{Z}/2\mathbf{Z} \times H) \sim S_n = \sum_{\langle\lambda\rangle} I_{\langle\lambda\rangle}$$

indexed by multipartitions. Then, Berele's theorem from [1] states that $Q_n(A|H)$ is contained in the sum of the ideals $I_{\langle\lambda\rangle}$ with all parts in the $k \times l$ hook. But, for every α in this ideal there exists a w in W_n such that $w\alpha^* \neq 0$ by [5]. So $w\alpha^* \notin I_n(A|H)$; and this can happen if and only if $w \notin I_n(A^*|H)$. Hence, if a multilinear, graded H-polynomial is not an identity for A^* then it has a specialization into k degree zero variables and l degree one variables which is not in $I_n(A|H)$, and so it is not an identity for $U_{k,l}(A^*)$. This completes the proof. ∎

3. GENERAL STRUCTURE

LEMMA 2. *Let A be a finitely generated algebra with H-action. Then the intersection of the H-prime ideals is nilpotent.*

Proof. Since A is a p.i. algebra, the maximal ideals of A are precisely the primitive ideals of A. In addition, since A is finitely generated, the Jacobson radical of A is nilpotent. Thus the intersection of the maximal ideals of A is nilpotent. If P is a maximal ideal and if Q is the largest H-stable ideal of A contained in P, then Q is an H-prime ideal. As a result, the intersection of the H-prime ideals is contained in the Jacobson radical and therefore must be nilpotent. ∎

LEMMA 3. *Let B be H-prime and $\mathbf{Z}/2\mathbf{Z}$-graded in Kemer's sense. Then B^* is H-verbally prime. Moreover, $B_0 \cap B_1$ is an H-stable ideal of B.*

Proof. Let I and J be (ungraded) H-T-ideals such that $I(B^*)$ and $J(B^*)$ are non-zero. Then $I^*(B)$ and $J^*(B)$ are non-zero H-ideals of B. So, their product I^*J^* does not vanish on B, hence IJ does not vanish on B^*. The second assertion is clear. ∎

THEOREM 1. *Let A be any p.i. algebra with H-action. Then A has a nilpotent H-ideal J such that A/J is H-verbally semiprime. Moreover, J is an intersection of H-verbally prime ideals, and every H-verbally prime ideal is the ideal of identities of some B^*, where B is H-prime and $B_0 \cap B_1$ is an H-stable ideal of B.*

Proof. Let B be a finitely generated algebra, graded H-p.i. equivalent to A^*. In B the intersection of the H-prime ideals $\cap I_\alpha$ is nilpotent. For each I_α let J_α be the identities of B/I_α. Then the evaluation $J_\alpha(B)$ is contained in I_α and so the intersection of the J_α will be nilpotent. But, by the previous lemma, A/J_α^* is verbally prime. The intersection will be a nilpotent ideal of A modulo which A will be verbally semiprime. ∎

If H is a group algebra or the dual of a group algebra, there is more we can say.

LEMMA 4. *Let A be a finitely generated H-prime p.i. algebra, where H is either the group algebra of a finite group or the dual of a group algebra of a finite abelian group. Also, let Z denote the center of A and let K be the quotient field of the integral domain Z^H. If $A_{(Z^H)}$ is the localization of A at the nonzero elements of Z^H, then $A_{(Z^H)}$ is semiprime, H-simple and finite-dimensional over the field K.*

Proof. Since A is H-prime and every element of Z^H generates an H-stable ideal of A, the nonzero elements of Z^H are regular in A. Furthermore, since H is a group algebra or its dual, A is also semiprime and it follows that every nonzero ideal of A intersects Z nontrivially. In addition, since H is cocommutative, Z is H-stable. However, since Z has no nilpotent elements and H is a group algebra or its dual, every nonzero H-stable ideal of Z intersects Z^H nontrivially. Thus every nonzero H-stable ideal of A intersects Z^H nontrivially and so, $A_{(Z^H)}$ is both semiprime and H-simple and the center of $A_{(Z^H)}$ is the quotient field of Z^H. Therefore, it suffices to consider the case where A is H-simple and Z^H is a field and we will need to show that A is finite-dimensional over Z^H.

To this end, we can form the smash product $Z\#H$ and let it act on Z. The simple $Z\#H$-submodules of Z are the H-stable ideals of Z. Since every nonzero H-stable ideal of Z intersects Z^H nontrivially and Z^H is a field, it follows that Z is a simple $Z\#H$-module.

Let M denote the annihilator of the action of $Z\#H$ on Z. Then Z is a faithful simple $(Z\#H)/M$-module. However, $Z\#H$ is a finite module over the commutative ring Z, thus $Z\#H$ satisfies a p.i. and clearly $(Z\#H)/M$ must also satisfy a p.i. As a result, $(Z\#H)/M$ is a primitive p.i. ring and therefore must be simple. Let $\int$ denote the integral of H; since $\int$ is the identity map on Z^H, it follows that $\int \notin M$. Therefore $(Z\int Z)/M$ is a nonzero ideal of $(Z\#H)/M$ and so, $(Z\int Z)/M = (Z\#H)/M$.

Therefore there exists $v \in Z \int Z$ and $w \in M$ such that

$$1 = v + w.$$

Note that v must be of the form $\sum_{i=1}^{m} a_i \int b_i$, where $a_i, b_i \in Z$. If $r \in Z$ and if we let both sides of the equation act on r we obtain

$$r = v \cdot r + w \cdot r = \sum_{i=1}^{m} a_i t(b_i r),$$

where t is the trace map. Now consider the map

$$\phi: Z \to \bigoplus_{i=i}^{m} Z^H$$

defined as

$$\phi(r) = \bigoplus_{i=i}^{m} t(b_i r).$$

Clearly ϕ is a linear transformation of vector spaces over the field Z^H. Furthermore, if $\phi(r) = 0$, then $t(b_i r) = 0$, for all $i \leq m$ and it follows that $r = \sum a_i t(b_i r) = 0$. Therefore ϕ is an injection and we can consider Z as a subspace of the m-dimensional space $(Z^H)^m$. Hence Z is l-dimensional over Z^H, for some $l \leq m$.

Suppose $B_1 \oplus \cdots \oplus B_s$ is a direct sum of ideals of A. Since each B_i intersects Z nontrivially, each B_i contains some nonzero $b_i \in Z$. Certainly the b_i are linearly independent over Z^H, hence $s \leq l$. Therefore there exists a positive integer n such that A contains a direct sum of n ideals and A contains no longer direct sum of ideals. Thus there exists ideals $C_1, \dots, C_n$ such that $C_1 \oplus \cdots \oplus C_n$ is a direct sum.

Each C_i must be a prime ring, for if D, E were nonzero ideals of C_i with $DE = 0$, then we could replace C_i in the direct sum by $C_i D C_i \oplus C_i E C_i$, thereby contracting the choice of n. If we let Z_i denote the center of C_i, then Z_i is a commutative domain which is contained in Z. Therefore Z_i is also finite-dimensional over the field Z^H and hence Z_i must also be a field. However, since C_i is a prime p.i. algebras whose center is a field, it follows that C_i is simple and finite-dimensional over Z^H. In light of this, the sum $C_1 \oplus \cdots \oplus C_n$ is also finite-dimensional over Z^H.

Finally, let e be the identity of $C_1 \oplus \cdots \oplus C_n$. Since $(1 - e)A \cap (C_1 \oplus \cdots \oplus C_n) = 0$, we could produce a longer direct sum unless $(1 - e)A = 0$. Thus $(1 - e)A = 0$ and so, e is the identity of A. Therefore $A = C_1 \oplus \cdots \oplus C_n$ and A is finite-dimensional over Z^H. ∎

COROLLARY. *Let A and H be as in the previous lemma and assume that K is algebraically closed. Then $A_{(Z^H)}$ is a finite direct sum of matrix algebras of the same size.*

Proof. Suppose $a_1, \ldots, a_n \in A_{Z^H}$, $h \in H$, and $\sigma \in S_n$. Then

$$h \cdot (a_{\sigma(1)} \cdots a_{\sigma(n)}) = \sum (h_1 \cdot a_{\sigma(1)}) \cdots (h_n \cdot a_{\sigma(n)}).$$

However, since H is cocommutative, we can permute the h_i to obtain

$$h \cdot (a_{\sigma(1)} \cdots a_{\sigma(n)}) = \sum (h_{\sigma(1)} \cdot a_{\sigma(1)}) \cdots (h_{\sigma(n)} \cdot a_{\sigma(n)}).$$

Therefore, we have

$$h \cdot s_n(a_1, \ldots, a_n) = s_n(h_{\sigma(1)} \cdot a_{\sigma(1)}, \ldots, h_{\sigma(n)} \cdot a_{\sigma(n)}).$$

As a result, the values of any standard identity generate an H-stable ideal of $A_{(Z^H)}$. Now let m be the smallest positive integer such that $M_m(K)$ appears as a direct summand in $A_{(Z^H)}$. Then the values of s_{2m} generate an H-stable ideal whose intersection with the direct summand $M_m(K)$ is zero. Since $A_{(Z^H)}$ is H-simple this implies that the ideal generated by the values of s_{2m} must be zero. Thus $A_{(Z^H)}$ satisfies s_{2m} and all the direct summands in $A_{(Z^H)}$ are the $m \times m$ matrices over K. ∎

4. GROUP ACTIONS

At this point one would like to classify all simple, graded algebras with an H-action and so classify all H-verbally prime algebras. In particular, it would be desirable to show that every H-semiprime ideal is a *finite* intersection of H-verbally prime ideals. Unfortunately, we don't know how to do the general case. In this section we consider algebras with action from a fixed, finite group G. We will show, that every G-verbally semiprime ideal is a finite intersection of G-verbally prime ideals and we describe G-verbally prime ideals.

Here are some useful lemmas on algebras with actions by finite groups.

LEMMA 5. *Let G be a finite group and A a simple G-algebra. Then A is a direct sum of isomorphic simple algebras.*

Proof. If A is not simple, let I be a maximal ideal. Then each $g(I)$, $g \in G$ will also be a maximal ideal and A/I and $A/g(I)$ will be isomorphic. Finally, the intersection $\bigcap_{g \in G} g(I)$ is a G-invariant ideal and so must be zero. ∎

LEMMA 6. *Let $A = M_n(F)^t$ be a direct sum of matrix algebras of the same size. Assume that A is simple as a G-algebra, and that A has a $Z/2Z$-grading (in Kemer's sense) which is compatible with the group action; i.e., $A = A_0 + A_1$, each $A_i A_j \subseteq A_{i+j}$ and each $G(A_i) \subseteq A_i$. Then either $A = A_0 = A_1$, or A is a direct sum of isomorphic graded simple ideals, each of which is $M_n(F)$ or $M_n(F) \oplus M_n(F)$.*

Proof. $A_0 \cap A_1$ is a G-stable ideal, hence it is either A or 0. In the former case, we are done. In the latter case we construct the involution that acts as the identity on A_0 and as multiplication by -1 on A_1. Then an ideal of A will be homogeneous if and only if it is stable under this involution. But the involution must permute the simple ideals. Let $I \cong M_n(F)$ be one such. Then $J = I + I^*$ will be a minimal invariant ideal. And $A = \sum_{g \in G} g(J)$ is the desired direct sum decomposition. ∎

DEFINITION. (a) Given an $n \times n$ matrix M over any ring and given $k \geq l \geq 0, k + l = n$ we decompose M as $\left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right)$, where A is a $k \times k$ submatrix, B is a $k \times l$ submatrix, C is an $l \times k$ submatrix, and D is an $l \times l$ submatrix. Then A and D will be called the (k, l) diagonal blocks and B and C will be called the (k, l) off diagonal blocks.

(b) Given any algebra R, let $M(k, l; R)$ denote $M_{k+l}(R)$ with a $\mathbf{Z}/2\mathbf{Z}$-grading in which the degree zero part consists of matrices which are zero in the (k, l) off diagonal blocks, and the degree one part consists of matrices which are zero in the (k, l) diagonal blocks. Note that if $l = 0$ then we get $M_k(R)$ concentrated in degree zero.

(c) Let $\widetilde{M}_n(R) = M_n(R) \oplus M_n(R)$ with a $\mathbf{Z}/2\mathbf{Z}$-grading in which the degree zero part consists of pairs of equal matrices and the degree one part consists of pairs of matrices which are negatives of each other.

Note that as a graded algebra $\tilde{M}_n(E)$ is isomorphic to $M_n(E)u$, where u is a central, degree one element with $u^2 = 1$ and with E considered to be ungraded.

LEMMA 7. (a) *If $k \neq l$ then the graded automorphisms of $M(k, l)$ consists of $((GL_k(F) \times GL_l(F))/F)$.*

(b) *The graded automorphisms of $M(k, k)$ consists of $((GL_k(F) \times GL_k(F))[g]/F)$, where we adjoin $g = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) \in GL_{2k}(F)$.*

(c) *The graded automorphisms of $\widetilde{M}_n(F)$ consists of $PGL_n(F)[g]$ where g acts as the identity on the degree zero part and as multiplication by -1 on the degree one part. Note that in this case g is central.*

Proof. Every isomorphism $M_n(F) \to M_n(F)$ must be given by conjugation, by the Skolem–Noether theorem. Now, for (a) and (b), note if this conjugation takes the degree zero part, $M_k(F) \oplus M_l(F)$ to itself then either it takes $M_k(F)$ to itself and $M_l(F)$ to itself, in which case it comes from $GL_k(F) \times GL_l(F)$; or it reverses the two, in which case $k = l$ and g times it comes from $GL_k(F) \times GL_k(F)$.

For (c), an automorphism of $\widetilde{M}_n(F)$ will act on the degree zero part by conjugation from $PGL_n(F)$, since the degree zero part is isomorphic to $M_n(F)$. And, since the action is multiplicative, the action on the degree one part will be determined by the image of $(I, -I)$, where I is the identity. But, $(I, -I)$ must go to an element which is central, degree one, and has square the identity, (I, I). There are only two such elements: $(I, -I)$ and $(-I, I)$. ∎

LEMMA 8. *Given an algebra A with $\mathbf{Z}/2\mathbf{Z}$-grading let* $\operatorname{Aut}(A)$ *be the group of graded automorphisms of A.*

(a) *Let $\phi_1, \phi_2\colon G \to \operatorname{Aut}(A)$ be group homomorphisms. Then ϕ_1 and ϕ_2 define isomorphic G actions on A if and only if $\phi_1(G)$ and $\phi_2(G)$ are conjugate in* $\operatorname{Aut}(A)$.

(b) *If $A = M_n(F) \oplus \cdots \oplus M_n(F) = M_n(F)^t$, then* $\operatorname{Aut}(A) \cong PGL_n(F) \sim S_t$, *i.e., for every automorphism $f\colon A \to A$ there exists $b_1, \dots, b_t$ invertible $n \times n$ matrices and a permutation $\sigma \in S_t$ such that $f(a_1, \dots, a_t) = (b_1 a_{\sigma(1)} b_1^{-1}, \dots, b_t a_{\sigma(t)} b_t^{-1})$, for all $(a_1, \dots, a_t) \in A$.*

Proof. (a) The two actions are isomorphic if and only if there exists a $\theta \in \operatorname{Aut}(A)$ such that $\theta(\phi_1(g)a) = \phi_2(g)\theta(a)$, for all $a \in A, g \in G$. But this is equivalent to $\theta\phi_1\theta^{-1} = \phi_2$.

(b) Given $g \in G$, g must permute the copies of $M_n(F)$ because they are minimal ideals. Moreover, any isomorphism $M_n(F) \to M_n(F)$ must be given by conjugation, by the Skolem–Noether theorem. ∎

We leave the proof of the next lemma to the reader.

LEMMA 9. *Let $A = M_n(F)^t$ be a simple G-algebra with a compatible $\mathbf{Z}/2\mathbf{Z}$-grading, as in Lemma* 6, *and let* $\operatorname{Aut}(A)$ *denote the graded automorphisms of A.*

(a) *If $A = A_0 = A_1$, then* $\operatorname{Aut}(A) = PGL_n(F) \sim S_t$.

(b) *If $A_0 \cap A_1 = 0$ and each copy of $M_n(F)$ is graded, then $A = M(k,l)^t$ and* $\operatorname{Aut}(A) = \operatorname{Aut} M(k,l) \sim S_t$, *where* $\operatorname{Aut} M(k,l)$ *is as in Lemma* 7.

(c) *If $A_0 \cap A_1 = 0$ and each copy of $M_n(F)$ is not graded, then $t = 2q$ is even, $A = \tilde{M}_n(F)^q$, and* $\operatorname{Aut}(A) = \operatorname{Aut} \widetilde{M}_n(F) \sim S_q = PGL_n(F) \sim H_q$, *where H_q is the octahedral group, $\mathbf{Z}/2\mathbf{Z} \sim S_q$.*

We now need that there are only finitely many embeddings of a given group G into $\operatorname{Aut}(A)$, up to conjugation in the endomorphism group.

Fortunately, this fact was recently proven by P. Slodowy in [6] and by V. Platonov and A. Rapinchuk in [4]. Slodowy says that the result is essentially due to Weil.

THEOREM (Weil–Platonov–Rapinchuk–Slodowy). *If G is a finite group and $\mathscr{G}$ is an algebraic group over an algebraically closed field, with characteristic zero or prime to $|G|$, then there are only finitely many embeddings of G into $\mathscr{G}$, up to conjugation in $\mathscr{G}$.*

In the case of finite group actions we may now strength the results of Section 3.

THEOREM 2. *Let A be a p.i. algebra with action from the finite group G. If A is G-verbally semiprime, then A is G-p.i. equivalent to a finite direct sum of G-verbally prime algebras, and if A is G-verbally prime, then A is G-p.i. equivalent to one of*

(1) $M_n(E)^t$ *for some n and t; or*

(2) $M_{k,l}^t$ *for some k, l, and t; or*

(3) $\widetilde{M}_n(E)^t$,

with group action as in Lemma 9.

Proof. As in Theorem 1, A is equivalent to B^*, where B is a direct sum of G-simple $\mathbf{Z}/2\mathbf{Z}$-graded algebras. We first show that, up to G-p.i. equivalence, there are only finitely many simple algebras C which satisfy all the identities of B. Tensoring with the algebraic closure of F we may assume by Lemma 5 that all these simple algebras are $n \times n$ matrices, for some n. Since C is G-simple, the simple algebras will be permuted by G, so there will be at most $|G|$ of them. Also, the size of the matrices is bounded by the degree of the standard identity satisfied by B which is in turn determined by the identities of A. Finally, by Lemma 9, the automorphism group of C is an algebraic group so we may apply the Weil–Platonov–Rapinchuk–Slodowy theorem to conclude that each $M_n(F)^t$ admits only finitely many G actions up to conjugation. This shows that up to graded H-p.i. equivalence there are only finitely many simple G-algebras which satisfy the identities of B. ∎

5. GROUP GRADINGS

If we try to generalize the material of the previous section to the case of an algebra A with grading by the finite group G, two problems arise. First of all, a graded simple algebra need not be a direct sum of simple isomorphic algebras. For example, if A is the group algebra FG, then A is graded simple, but in general it will be a direct sum of matrix algebras of different sizes. Second, we don't have an analog of the Weil–Platonov–Rapinchuk–Slodowy theorem to tell us that there are only finitely many G-gradings on a given algebra, up to conjugation. We have not been able to prove that a G-verbally semiprime algebra is equivalent to a finite direct sum of G-verbally prime algebras. The question seems to be an interesting one.

LEMMA 10. *Let G be a finite group and A a G-graded simple algebra. Then A is p.i. equivalent to a direct sum of at most $|G|$ simple algebras.*

Proof. By Theorem 6.3 of [2], A contains at most $|G|$ minimal primes $P_1, P_2, \ldots, P_m$ and $\bigcap_{i=1}^{m} P_i = 0$. Therefore A is p.i.-equivalent to the direct sum $A/P_1 \oplus \cdots \oplus A/P_m$. Each A/P_i is a prime p.i. algebra and is certainly p.i. equivalent to a simple algebra. ∎

Arguing as in the previous section we get

COROLLARY. *Every G-verbally prime algebra is G-p.i. equivalent to an algebra which is a direct sum of at most $|G|$ algebras of the form $M_n(E)$ or $M_{k,l}$.*

If G is abelian we may use the duality between group actions and group gradings to obtain analogs of all of the results of the previous section.

THEOREM 3. *Let G be a finite abelian group and A a G-graded, G-verbally semiprime p.i. algebra. Then A is G-p.i. equivalent to a finite direct sum of G-verbally prime algebras. Moreover, every G-verbally prime algebra is equivalent to an algebra obtained from an algebra in Theorem 2 by changing the group action into a grading using duality.*

Proof. If G is a finite abelian group, let $\hat{G} = \mathrm{Hom}(G, F^x)$. Since F contains all roots of unity, $G \cong \hat{G}$. The grading of A by G corresponds to an action of A by $\hat{G}$ as automorphisms via $\lambda(r_g) = \lambda(g) r_g$, for all $\lambda \in \hat{G}, g \in G$. When we consider the elements of G and $\hat{G}$ as F-linear transformations of A, every $g \in G$ is an F-linear combination of elements of $\hat{G}$. Therefore every G-p.i. for A corresponds to a $\hat{G}$-p.i. As a result, we can now apply Theorem 2. ∎

6. EXAMPLES

In this section we apply the results of Section 4 to classify p.i. algebras with action from a cyclic group $G = \mathbf{Z}/m\mathbf{Z}$ up to G-p.i. equivalence. As a corollary we will be able to classify p.i. algebras with $\mathbf{Z}/2\mathbf{Z}$-grading. This is probably of interest, since these algebras play an important role in Kemer's work.

By Theorem 2 we have three cases to consider. In each case $A = B^*$ where B is a direct sum of matrix algebras. If G acts transitively on the simple ideals then B^* will be a simple G-algebra. Our next three lemmas discuss which G actions are possible, up to conjugation.

LEMMA 11. *Let $B = M_n(F)^t$ be a simple $G = \mathbf{Z}/m\mathbf{Z}$-algebra and let $\alpha \in GL_n(F) \sim S_t$ be a generator for the image of G in* $\operatorname{Aut}(B)$. *Then t divides k and α is conjugate to an element of the form $(I, I, \ldots, I, \Omega; (12 \ldots t))$, where I is the identity matrix and Ω is a diagonal matrix with $\Omega^{m/t} = 1$.*

Proof. Since B is simple, α will act transitively on the simple ideals of A. It follows that t divides m and that the permutation is a t-cycle. By conjugation, we may assume that it acts as $(12 \ldots t)$. Now, if α acts as $(b_1, \ldots, b_t; (12 \ldots t))$, then $\alpha^m = 1$ implies that $(b_1 \cdots b_t)^{m/t} = 1$. Now calculate

$$\begin{aligned}(c_1, \ldots, c_t; 1)\big(b_1, \ldots, b_t; (12 \ldots t)\big(c_1^{-1}, \ldots, c_t^{-1}; 1)\big) \\ = \big(c_1 b_1 c_2^{-1}, c_2 b_2 c_3^{-1}, \ldots, c_t b_t c_1^{-1}; (12 \ldots t)\big).\end{aligned}$$

Hence, if we take each $c_i = b_1 \cdots b_{i-1}$ we get

$$\big(I, I, \ldots, I, b_1 b_2 \cdots b_t; (12 \ldots t)\big).$$

We may assume that the field F contains all the necessary roots of 1, and so we may assume that $(b_1 \cdots b_t)$ can be conjugated, say by c, to a diagonal matrix with roots of 1 on the diagonal. Now conjugating the above by $(c, \ldots, c; 1)$ gives the desired form. ∎

LEMMA 12. *Let $B = M(k, l)^t$ have an action from $G = \mathbf{Z}/m\mathbf{Z}$ which preserves degree and such that B is simple as a G-algebra. Let $\alpha \in GL_n(F) \sim S_t$ be a generator for the image of G in* $\operatorname{Aut}(B)$. *Then t divides m and either α is as in the previous lemma, or $k = l$ and $\alpha = \Omega g$, where g is as in Lemma* 7(b), *with all conjugations preserving degree.*

Proof. In the proof of the previous lemma conjugation by each b_i will preserve degree and therefore so will conjugation by any product $b_1 \cdots b_{i-1}$. As for c, if $b_1 \cdots b_t$ preserves degree we may take c to preserve degree, and if it reverses degree in the $k = l$ case we may take c to reverse degree and so conjugation by c will preserve degree. ∎

For the case of $\widetilde{M}_n(F)$, note that the graded endomorphism group is $PGL_n(F) \times \mathbf{Z}/2\mathbf{Z}$, so the automorphism group of $\widetilde{M}_n(F)^t$ will be $(PGL_n(F) \times \mathbf{Z}/2\mathbf{Z}) \sim S_t$ which is isomorphic to $PGL_n(F) \sim H_t$ where $H_t \subseteq S_{2t}$ is the hyperoctahedral group $\mathbf{Z}/2\mathbf{Z} \sim S_t$.

LEMMA 13. *Let $B = M_n(F)^{2t} = (\widetilde{M}_n(F))^t$ as a $\mathbf{Z}/2\mathbf{Z}$-graded algebra. Let B have an action from $G = \mathbf{Z}/m\mathbf{Z}$ which preserves degree and such that B is simple as a G-algebra. Let $\alpha \in GL_n(F) \sim S_{2t}$ be a generator for the image of G in* $\mathrm{Aut}(B)$. *Then $2t$ divides m and α is conjugate to an element of the form* $(I, I, \dots, I, \Omega; (12)(34) \cdots (2t-1, 2t)(135 \dots 2t-1))$.

Proof. As in the previous cases, α acts transitively on the simple ideals and so $2t$ divides m. Moreover, if we write $B = \oplus_i B_i$ as a sum of simple graded ideals then α will also act transitively on the B_i. By conjugation, we may assume that each B_i is sent to B_{i+1}. Also, within each $B_i \cong M_n(F) \oplus M_n(F)$, α must switch the two factors or it wouldn't be a $2t$-cycle. The rest of the proof is the same as that of Lemma 10. ∎

THEOREM 4. *Let $G = \mathbf{Z}/m\mathbf{Z}$. Then every G-verbally prime p.i. algebra is G-p.i. equivalent to some A^t, t divides m; where A is one of $M_n(F)$, $M_{k,l}$ or $M_n(E)$; and where a generator α of G acts by sending $(A_1, \dots, A_t)$ to $(A_2, \dots, A_t, \Omega A_1 \Omega^{-1})$ where Ω is a diagonal matrix whose (k/t)th power is the identity. There are two additional possibilities: In case of $M_{k,k}$, α could send $(A_1, \dots, A_t)$ to $(A_2, \dots, A_t, (\Omega g) A_1 (\Omega g)^{-1})$, where g is as in Lemma* 7(b) *and where $(\Omega g)^k = 1$; or in the case of $A = M_n(E)$ and t even, α could send $(A_1, \dots, A_t)$ to $(\overline{A_2}, \dots, \overline{A_t}, \Omega \overline{A_1} \Omega^{-1})$, the bar denoting the involution obtained from the natural involution on E.*

Proof. We calculate B^* in each of the three preceding lemmas. Lemma 11 gives $M_n(E)^t$ and Lemma 12 gives $M_{k,l}^t$ and $M_n(F)^t$, in the case of $l = 0$. In Lemma 13, B^* is $M_n(E)^t$ and the action by $(12)(34) \cdots$ gives the conjugation action in E. ∎

Specializing to $m = 2$ and using duality between group actions and gradings determine the verbally prime $\mathbf{Z}/2\mathbf{Z}$-graded p.i. algebras. We leave the details to the reader.

THEOREM 5. *Every $\mathbf{Z}/2\mathbf{Z}$-graded verbally prime p.i. algebra is graded p.i. equivalent to one of the following:*

(a) $\widetilde{M}_n(E)$;

(b) *$M_n(E) \oplus M_n(E)$ with degree zero part consisting of all pairs $(A, \overline{A})$ and degree one part consisting of all $(A, -\overline{A})$, where the bar denotes the natural involution in E;*

(c) $\widetilde{M}_{k,l}$;

(d) $M(k, l; F)$;

(e) $M(k, l; E)$;

(f) *Write $n = n_{00} + n_{01} + n_{10} + n_{11}$. We define the algebra $M(n(i, j))$ as the set of $n \times n$ matrices M with entries from $E_0 \cup E_1$, the homogeneous* *elements of* E, *decomposed into* 16 *rectangular blocks* $M = (M_{ab,cd})$ *of dimensions* $n_{ab} \times n_{cd}$, *such that* $M_{ab,cd}$ *will have entries from* E_{b+d} *and will be of degree* $a + c$. (*Note that this case includes* $M_{k,l}$ *with its usual grading and with the trivial grading*.);

(g) $M_{k,k}$ *with degree zero part all* $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ *and degree one part all* $\begin{pmatrix} A & B \\ -B & -A \end{pmatrix}$;

(h) $M_{k+l}(E)$, *in which the degree zero elements have even elements of* E *on the* (k,l) *diagonal blocks and odd elements of* E *on the off diagonal blocks*; *and the degree one part is the opposite*. (*Note that if* $l = 0$ *this gives* $M_n(E)$ *with grading induced by the natural grading on* E.)

REFERENCES

1. A. Berele, Cocharacter sequences for algebras with Hopf algebra action, *J. Algebra* **185** (1996).
2. M. Cohen and S. Montgomery, Group-graded rings, smash products and group actions, *Trans. Amer. Math. Soc.* **282** (1984).
3. A. R. Kemer, Varieties and Z_2-graded algebras, *Math. USSR Izv.* **25** (1985).
4. V. Platonov and A. Rapinchuk, ‘‘Algebraic Groups and Number Theory,’’ Academic Press, San Diego, 1994.
5. A. Regev, The representations of wreath products via double centralizer theorems, *J. Algebra* **102** (1986).
6. P. Slodowy, Two notes on a finiteness problem in the representation theory of finite groups, *in* ‘‘Algebraic Groups and Lie Groups’’ (G. I. Lehrer, Ed.), Cambridge Univ. Press, Cambridge, UK, 1997.